The 8th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2022
5-9 June 2022, Oslo, Norway

# Numerical analysis of some nonlinear hyperbolic systems of Partial Differential Equations arising from Fluid Mechanics

**Ernesto Pimentel-García**[1,*]

[1] University of Málaga, Facultad de Ciencias, Campus de Teatinos s/n, 29071, Málaga, España. E-mail: erpigar@uma.es

**Keywords**: *Hyperbolic Equations, Riemann Problems, Finite Volume Methods, Well-Balanced Methods, Non-Conservative Products*

This thesis addresses four different problems related to the numerical analysis of hyperbolic systems of nonlinear partial differential equations: the uniqueness of solution, the computational effort of numerical methods, the well-balanced property and the control of numerical viscosity. These problems are related to some of the lines of research of the EDANYA group and, more specifically, to the numerical resolution of mathematical models of fluid mechanics in applications related to shallow water flows and gas dynamics in the context of classical or relativistic mechanics. The issues addressed are listed below in chronological order.

The first problem addressed is the study of the Riemann problem for the shallow water equations on a step-shaped bottom, in the particular case in which there is only water on one side of the step. In this way we will complete the study carried out by LeFloch and Thanh in [1]. The analysis of these situations of dry-wet fronts is important when designing numerical schemes that deal well with flood phenomena. Two major difficulties arise when studying this problem. On the one hand, the source term that appears in the equations is a nonconservative product, so that there is no a unique way to define the weak solutions to the problem. In our case we will follow the theory of Dal Maso, LeFloch and Murat [2] to define them from a family of paths. On the other hand, resonant cases appear (i.e., an eigenvalue of the Jacobian matrix vanishes) that implies that, once the definition of weak solution is chosen, there is no uniqueness of solution. This theoretical study is complemented with numerical tests in which the behavior of different schemes is studied. The content of this work was published in [3].

The next question to be addressed is the efficient implementation of numerical methods based on approximate Riemann solvers and, in particular, the Roe method, which is based on solving linearized Riemann problems in the intercells. The practical interest of this chapter resides above all in the numerical resolution of large systems such as multilayer shallow water models [4] or models based on moments [5]. This new implementation is based on the close relationship between this kind of methods and the Polynomial Viscosity Matrix ones based on the election of a polynomial that interpolates the absolute value function, as well as on the Newtonian form of the interpolation polynomial, which is the most efficient one. It will be seen that the computational cost reduction grows with the number of equations. The content of this work was published in [6].

The next objective is to make a systematic study of the asymptotic behavior of the solutions of the relativistic Burgers and Euler models based on the Schwarzchild metric, using numerical methods. This metric is an exact solution of the Einstein equations of the gravitational field that describes the field generated by a star or a spherical mass. In these systems the stationary solutions and the evolution of its perturbations play a fundamental role in understanding the flow behavior. Therefore, the use of well-balanced methods, i.e., methods that preserve these solutions, is essential. We will apply the general framework described in [7] to develop well-balanced methods of order up to 3 for the Burgers-Schwarzschild model and 2 for the Euler-Schwarzschild model. In the case of the relativistic Burgers model, it is possible to compute its stationary solutions explicitly, which will allow us to apply directly the well-balanced reconstruction procedure. In the case of the relativistic Euler equations, the implicit expression of the stationary solutions is available, so that a numerical method is necessary to evaluate them at a point: we will use the Newton's method. We will compare the results obtained between these methods and the standard ones and we will show the relevance of the well-balanced property. The content of this work was published in [8].

It is well-known that, in the case of systems with nonconservative products, consistency, stability and entropy control are not sufficient to ensure the convergence of the numerical approximations to admissible weak solutions: it is also necessary to control the small-scale effects such as the numerical viscosity that affects the position and amplitude of shock waves (see [9]). In [10] Chalons presented a technique based on in-cell discontinuous reconstructions that allowed him to design first-order methods that capture exactly isolated shocks. The last problem that arises in the thesis is the extension to second-order of this technique using the formalism of the path-conservative methods introduced in [11], what will allow to set the basis for its generalization to arbitrary order. We will state and prove that this second-order extension keeps the property of capturing exactly isolated shocks and we will verify this through different numerical tests, improving the results obtained with the method introduced in [10]. The content of this work was published in [12].

Throughout this thesis we have considered a wide number of analytical and numerical aspects of conservative and nonconservative hyperbolic systems of Partial Differential Equations. Different problems that arise when dealing with this kind of systems have been studied and solved. In the last part of the thesis we have highlighted the main contributions in each topic and the possible future works that arise from this dissertation.

## REFERENCES


[1] G. LeFloch and M. D. Thanh. *The Riemann problem for the shallow water equations with discontinuous topography.* Communications in Mathematical Sciences, 5(4):865–885, 2007.

[2] G. Dal Maso, P. G. LeFloch, and F. Murat. *Definition and weak stability of non- conservative products.* Journal de Mathématiques Pures et Appliquées, 74(6):483–548, 1995.

[3] C. Parés and E. Pimentel-García. *The Riemann problem for the shallow water equations with discontinuous topography: The wet–dry case*. Journal of Computational Physics, 378:344–365, 2019.

[4] M. J. Castro, J. Macías, and C. Parés. *A Q-scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system*. ESAIM: Mathematical Modelling and Numerical Analysis, 35(1):107–127, 2001.

[5] J. Koellermeier. *Derivation and numerical solution of hyperbolic moment equations for rarefied gas flows*. Dissertation, RWTH Aachen University, Aachen, 2017.

[6] E. Pimentel-García, C. Parés, M. J. Castro, and J. Koellermeier. *On the efficient implementation of PVM methods and simple Riemann solvers. Application to the Roe method for large hyperbolic systems*. Applied Mathematics and Computation, 388:125544, 2020.

[7] M.J. Castro and C. Parés, *Well-balanced high-order finite volume methods for systems of balance laws*. Journal of Scientific Computing, 82(2), 1-48, 2020.

[8] P. G. Lefloch, C. Par és, and E. Pimentel-García. *A class of well-balanced algorithms for relativistic fluids on a schwarzschild background*. Journal of Scientific Computing, 89(1):1–43, 2021.

[9] P. G. LeFloch and S. Mishra. *Numerical methods with controlled dissipation for small-scale dependent shocks*. Acta Numerica, 23:743–816, 2014.

[10] C. Chalons. *Path-conservative in-cell discontinuous reconstruction schemes for non conservative hyperbolic systems*. Communications in Mathematical Sciences, 18(1):1–30, 2020.

[11] C. Parés. Numerical methods for nonconservative hyperbolic systems: a theoretical framework. SIAM Journal on Numerical Analysis, 44(1):300–321, 2006.

[12] E. Pimentel-García, M. J. Castro, C. Chalons, T. Morales de Luna, and C. Parés. *In-cell discontinuous reconstruction path-conservative methods for non conservative hyperbolic systems-Second-order extension*. Journal of Computational Physics, 459, 111152, 2022.